4.3 - 2nd Derivative of Natural Logarithm

Objectives

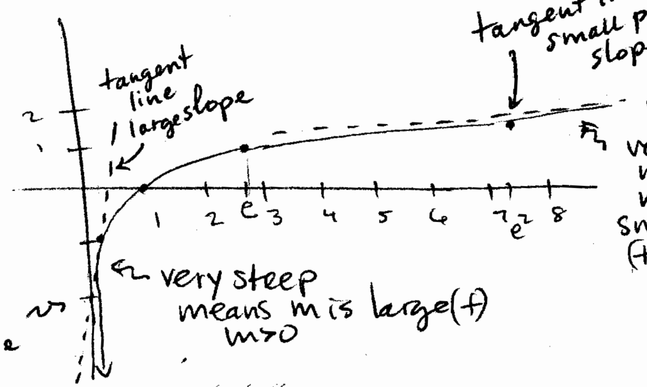
1) Find derivatives of $f(x) = \ln(x)$
2) Find derivatives using
   - → chain rule
   - → product rule
   - → quotient rule
   - → all these in combination with e^x
   - → simplify using log properties
   - → when you CAN'T simplify using log properties.

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$* \quad \frac{d}{dx}\ln(f(x)) = \underbrace{\frac{1}{f(x)}}_{\text{derivative of } \ln} \cdot \underbrace{f'(x)}_{\text{chain rule - derivative of inside}} = \frac{f'(x)}{f(x)} \quad *$$

Exploration: Given the graph of $f(x) = \ln(x)$, what characteristics of the derivative must we have?

Graph of $y = \ln x$



Vertical asymptote (GC lies! ☹)

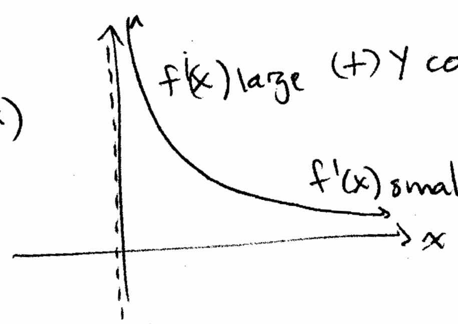
$\ln 0$ undef

so $f'(0)$ undefined!

all slopes of tangent lines are (+).

So $f'(x) > 0$ for all x in domain of $f(x) = \ln(x)$.

Graph of $f'(x)$



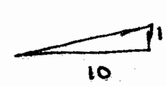
$$f'(x) = \frac{1}{x}!$$

$f'(.1)$ should be large

$= \frac{1}{.1} = 10$

$f'(10)$ should be small

$= \frac{1}{10}$



① Find equation of tangent line to $f(x) = \ln(x)$ at $x=1$
Graph $f(x) = \ln(x)$ and this tangent line in GC. $[-1,5] \times [-5,5]$

$f'(x) = \frac{1}{x}$

$f'(1) = \frac{1}{1} = 1$ = m of tangent line

$f(1) = \ln(1) = 0$ (1,0) point on $f(x)$ and tangent line

$y - y_1 = m(x - x_1)$
$y - 0 = 1(x-1)$
$y = x - 1$

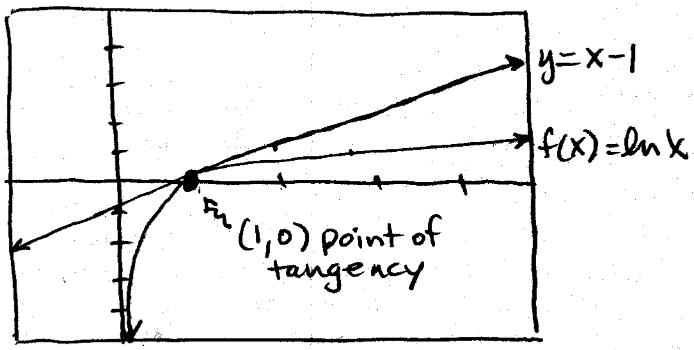
$y = mx + b$
$0 = 1(1) + b$
$-1 = b$
$y = x - 1$

$\boxed{GC}$

$\boxed{Y=}$
$Y_1 = \ln(x)$
$Y_2 = x - 1$

WINDOW

XMIN = -1
XMAX = 5
XSCL = 1
YMIN = -5
YMAX = 5
YSCL = 1



Find derivatives

(2) $f(x) = \ln(x^4-1)^3$

Easier: Use log properties before simplifying.

$$f(x) = 3\ln(x^4-1)$$

$$f'(x) = 3 \cdot \frac{1}{x^4-1} \cdot 4x^3$$

$$\boxed{f'(x) = \frac{12x^3}{x^4-1}}$$

Harder: double chain rule

$f(x) = \ln(x^4-1)^3$

outermost: ln
middle: (stuff)^3
inner: (x^4-1)

$$f'(x) = \frac{1}{(x^4-1)^3} \cdot \frac{d}{dx}(x^4-1)^3$$

derivative of inside

derivative of ln

$$= \frac{1}{(x^4-1)^3} \cdot 3(x^4-1)^2 \cdot \frac{d}{dx}(x^4-1)$$

derivative of middle = (stuff)^3

$$= \frac{1}{(x^4-1)^3} \cdot 3(x^4-1)^2 \cdot 4x^3$$

derivative of innermost

$$= \frac{12x^3(x^4-1)^2}{(x^4-1)^3}$$

exponent laws: subtract exp.

$$= \boxed{\frac{12x^3}{(x^4-1)^2}}$$

③ $f(x) = \ln(1+e^x)$

outer inner

No LOG PROPERTY for $\log_b(a \pm c)$

$$f'(x) = \frac{1}{1+e^x} \cdot \frac{d}{dx}(1+e^x)$$

$$= \frac{1}{1+e^x} \cdot (0+e^x)$$

$$\boxed{f'(x) = \frac{e^x}{1+e^x}}$$

$\log_b(a \cdot c) = \log_b a + \log_b c$

$\log_b\left(\frac{a}{c}\right) = \log_b a - \log_b c$

arguments multiplied or divided have properties

BUT NOT

arguments added or subtracted

④ $f(x) = \frac{\ln x}{x^3}$

x^3 is outside the log not part of argument.

CAN NOT SIMPLIFY.

Option 1: Quotient Rule as written

Option 2: Product Rule rewritten as negative exponent.

Your choice!

Option 1 $f'(x) = \frac{x^3 \cdot \frac{d}{dx}(\ln x) - \ln x \cdot \frac{d}{dx}(x^3)}{(x^3)^2}$ Quotient Rule

$$= \frac{x^3 \cdot \frac{1}{x} - \ln(x) \cdot 3x^2}{x^6}$$

subtract exponent

$$= \frac{x^2 - 3x^2 \ln x}{x^6}$$

→

$$= \frac{x^2(1-3\ln x)}{x^6}$$ factor GCF / least powers

$$= \boxed{\frac{1-3\ln x}{x^4}}$$ subtract exp

either ok

$$= \boxed{\frac{1-\ln(x^3)}{x^4}}$$ log property

Option 2:

$f(x) = x^{-3} \cdot \ln x$ rewrite with negative exp.

$f'(x) = \frac{d}{dx}(x^{-3}) \cdot \ln x + \frac{d}{dx}(\ln x) \cdot x^{-3}$ Product rule

$= -3x^{-4} \cdot \ln(x) + \frac{1}{x} \cdot x^{-3}$ power rule → subtract 1

$= -3x^{-4}\ln(x) + x^{-4}$ subtract exp $-3-1=-4$

$= \boxed{x^{-4}(-3\ln(x) + 1)}$ * factor least powers * (or GCF)

$= \boxed{\frac{1-3\ln(x)}{x^4}}$ as before

⑤ $y^2 - x\ln y = 10$

Must use implicit differentiation. ⇒ $\frac{dy}{dx}$ after each derivative of y.

$2y\frac{dy}{dx} - \left[x \cdot \frac{d}{dx}[\ln y] + \frac{d}{dx}[x] \cdot \ln y\right] = 0$ Product Rule!

$2y\frac{dy}{dx} - \left[x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + 1 \cdot \ln y\right] = 0$

$2y\frac{dy}{dx} - \frac{x}{y}\frac{dy}{dx} - \ln y = 0$ dist neg

$2y\frac{dy}{dx} - \frac{x}{y}\frac{dy}{dx} = \ln y$ collect $\frac{dy}{dx}$ terms

$\frac{dy}{dx}\left(2y - \frac{x}{y}\right) = \ln y$ factor out $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\ln y}{\left(2y - \frac{x}{y}\right)}$$

isolate $\frac{dy}{dx}$

fraction inside a fraction
= complex fraction
= not simplified ☹

To simplify complex fraction

Option 1: mult by $\frac{LCD}{LCD}$

$$\frac{dy}{dx} = \frac{\frac{\ln(y)}{1}}{\frac{2y}{1} - \frac{x}{y}}$$

denom y = LCD

$$= \frac{\frac{\ln(y)}{1} \cdot \frac{y}{1}}{\frac{2y}{1} \cdot \frac{y}{1} - \frac{x}{\cancel{y}} \cdot \frac{\cancel{y}}{1}}$$

mult all terms by $\frac{LCD}{1}$

(top & bottom makes mult by 1)

$$= \boxed{\frac{y \cdot \ln(y)}{2y^2 - x}}$$

Option 2: rewrite denom with CD, then mult by reciprocal

$$\frac{dy}{dx} = \frac{\ln(y)}{2y \cdot \frac{y}{y} - \frac{x}{y}}$$

mult by 1!

$$= \frac{\ln(y)}{\left(\frac{2y^2 - x}{y}\right)}$$

÷ fractions means mult by reciprocal

$$= \ln(y) \cdot \frac{y}{2y^2 - x}$$

$$= \boxed{\frac{y \cdot \ln(y)}{2y^2 - x}}$$

⑥ $f(x) = \ln(\underbrace{e^x - 2x})$

outer   inner

No log property for $\log_b(a - c)$!

see page 3 again! Problem ③ sidebar.

Two CHAIN RULES.

$$f'(x) = \frac{1}{e^x - 2x} \cdot \frac{d}{dx}(e^x - 2x)$$

$$= \frac{1}{e^x - 2x} \cdot (e^x - 2) = \boxed{\frac{e^x - 2}{e^x - 2x}}$$

⑦ $f(x) = \ln(e^x)$

Simplify

$f(x) = x$

$\boxed{f'(x) = 1}$

$\log_b b^x = x$ inverse property for logs and exponentials; base $b=e$ here

Asleep? CHAIN RULE.

$$f'(x) = \underbrace{\frac{1}{e^x}}_{\text{outside}} \cdot \underbrace{\frac{d}{dx}(e^x)}_{\text{inside}}$$

$$= \frac{1}{e^x} \cdot e^x$$

$$= \frac{e^x}{e^x}$$

$$= \boxed{1}.$$

⑧ $f(x) = e^x \underbrace{\ln(x^3)}_{\text{log property!}}$

$\log_b a^k = k \cdot \log_b a$

Simplify

$f(x) = 3 \cdot \underbrace{e^x \cdot \ln x}_{\text{product rule}}$

$$f'(x) = \underbrace{3}_{\text{constant multiple}}\left[e^x \cdot \frac{d}{dx}(\ln x) + \frac{d}{dx}(e^x) \cdot \ln x\right]$$

$$= 3\left[e^x \cdot \frac{1}{x} + e^x \cdot \ln x\right]$$

$$= \boxed{3e^x\left(\frac{1}{x} + \ln x\right)}$$ factor GCF

ASLEEP? PRODUCT AND CHAIN RULE...

$f(x) = e^x \cdot \ln x^3$

$f'(x) = \frac{d}{dx}(e^x) \cdot \ln(x^3) + e^x \cdot \frac{d}{dx}(\ln(x^3))$

$= e^x \cdot \ln(x^3) + e^x \cdot \frac{1}{x^3} \cdot \underbrace{\frac{d}{dx}(x^3)}$

chain rule inside

$= e^x \cdot \ln(x^3) + \frac{e^x}{x^3} \cdot 3x^2$ simplify exponents.

$= e^x \cdot \ln(x^3) + \frac{3e^x}{x}$ factor GCF

$= \boxed{e^x\left(\ln(x^3) + \frac{3}{x}\right)}$

if use log property now...

$= e^x\left(3\ln(x) + \frac{3}{x}\right)$ GCF 3 can be factored out.

$= \boxed{3e^x\left(\ln(x) + \frac{1}{x}\right)}$ Same as before

9. $f(x) = \ln\left(\frac{1}{e^{x^2}}\right)$

use log properties! (ALGEBRA!) $\log_b\left(\frac{a}{c}\right) = \log_b a - \log_b c$

$f(x) = \ln(1) - \ln(e^{x^2})$ $\log_b(1) = 0$ for any base b because $b^0 = 1$.

$f(x) = 0 - x^2$

$f(x) = -x^2$

inverse property $\log_b b^x = x$ so $\log_e e^{x^2} = x^2$

Now differentiate....

$\boxed{f'(x) = -2x}$

10. $f(x) = x^2e^x + x^2\ln x - 3\ln x - e^{x^2} + (x^3+2)^4 - \frac{1}{2}x^5 + \frac{1}{5}$

product, product, const multiple, chain, chain, power, constant

$f'(x) = \left[x^2 \cdot \frac{d}{dx}(e^x) + \frac{d}{dx}(x^2) \cdot e^x\right] + \left[x^2 \cdot \frac{d}{dx}(\ln x) + \frac{d}{dx}(x^2) \cdot \ln x\right] - 3 \cdot \frac{1}{x} - e^{x^2} \cdot \frac{d}{dx}(x^2) + \ldots$

$$+\ 4(x^3+2)^3 \cdot \frac{d}{dx}(x^3+2) - \frac{1}{2}\cdot 5x^4 + 0$$

↑ inside

$$f'(x) = x^2\cdot e^x + 2xe^x + x^2\cdot\frac{1}{x} + 2x\cdot \ln x - \frac{3}{x} - e^{x^2}\cdot 2x \cdots$$

$$+\ 4(x^3+2)^3\cdot(3x^2) - \frac{5}{2}x^4$$

$$\boxed{f'(x) = x^2e^x + 2xe^x + x + 2x\ln x - \frac{3}{x} - 2xe^{x^2} + 12x^2(x^3+2) - \frac{5}{2}x^4}$$

There are no like terms....
There is no GCF or least powers...

$$\boxed{f'(x) = x^2e^x + 2xe^x - 2xe^{x^2} + 2x\ln x + 12x^5 - \frac{5}{2}x^4 + 24x^2 + x - \frac{3}{x}}$$

dist $12x^2$

↑ e^x stuff first ↑ ln stuff next ↑ powers of x in descending order

(11) Use calculus to graph $f(x) = e^{-x^2}$ → critical values, increase/decrease, relative extrema, concave up/down, inflection points.

domain: all real numbers

range: $y>0$ for any exponential

$$f(x) = e^{-x^2} = \frac{1}{e^{x^2}} = \frac{1}{(+)} = (+)$$

$$f'(x) = e^{-x^2}\cdot\frac{d}{dx}(-x^2)$$

$$= e^{-x^2}(-2x)$$

$$\underline{\underline{f'(x) = -2xe^{-x^2}}} = \frac{-2x}{e^{x^2}}$$

$f''(x)$ product rule!

$$f''(x) = -2x\cdot\frac{d}{dx}(e^{-x^2}) + \frac{d}{dx}(-2x)\cdot e^{-x^2}$$

$$f''(x) = -2x\cdot e^{-x^2}\cdot(-2x) + (-2)e^{-x^2} \quad \ldots$$

$f''(x) = 4x^2e^{-x^2} - 2e^{-x^2}$ factor GCF

$\underline{\underline{f''(x) = 2e^{-x^2}(2x^2-1)}}$

critical values $f'(x)=0$

$f'(x) = \dfrac{-2x}{e^{x^2}} = 0$ mult both sides by e^{x^2}

$-2x = 0$

$\underline{\underline{x=0}}$

critical values $f'(x)$ undefined: (denominator $f'(x) = 0$)

$e^{x^2} = 0$

no solution because $b^{\text{stuff}} > 0$ for any (+) base b.

increase/decrease (sign chart for $f'(x)$)

f' (+) | (−)

0

max

Test $x<0$ $(-2)\cdot(-)\cdot(e^{\text{stuff}})$

$(-)\cdot(-)\cdot(+) = (+)$

Test $x>0$ $(-2)\cdot(+)\cdot(+)$

$(-)\cdot(+)(+) = (-)$

<u>increasing $(-\infty, 0)$</u>

<u>decreasing $(0, \infty)$</u>

relative max at $x=0$

$f(0) = e^{-0^2} = e^{-0} = e^0 = 1$

<u>relative max $(0,1)$</u>

$f''(x)=0$ $2e^{-x^2}(2x^2-1)=0$

$\dfrac{2(2x^2-1)}{e^{x^2}} = 0$ write with (+) exponent

$2(2x^2-1)=0$ mult both sides by e^{x^2} to clear fraction

$2x^2-1=0$ divide both sides by 2

$2x^2=1$

$x^2=\frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2} = x$ f''

$f''(x)$ undefined

$$\frac{2(2x^2-1)}{e^{x^2}} = 0$$

$\uparrow$ denom $= 0$

$$e^{x^2} = 0$$

no solution because $b^{\text{stuff}} > 0$ for base $b > 0$.

Concavity & Inflection (sign chart for f'')

f'': $(+)$ | $-\frac{1}{\sqrt{2}}$ | $(-)$ | $\frac{1}{\sqrt{2}}$ | $(+)$

$-\frac{1}{\sqrt{2}} \approx -.7$ $\quad$ $\frac{1}{\sqrt{2}} \approx +.7$

test $x = -2$
$x = 0$
$x = 2$

concave up

$$\left(-\infty, -\tfrac{1}{\sqrt{2}}\right); \left(\tfrac{1}{\sqrt{2}}, \infty\right)$$

concave down

$$\left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)$$

$$f''(-2) = \frac{2(2(-2)^2-1)}{e^{(-2)^2}} = \frac{(+)[8-1]}{(+)} = \frac{(+)(+)}{(+)} = (+)$$

$$f''(0) = \frac{(+)(2\cdot 0^2 - 1)}{(+)} = \frac{(+)(-)}{(+)} = (-)$$

$$f''(2) = \frac{(+)(2\cdot 2^2 - 1)}{(+)} = \frac{(+)(+)}{(+)} = (+)$$

inflection points

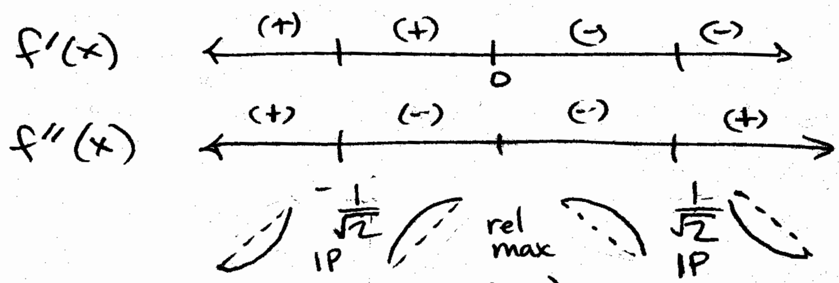
$$f\left(\tfrac{1}{\sqrt{2}}\right) = e^{-\left(\frac{\sqrt{2}}{2}\right)^2} = e^{-\left(\frac{1}{\sqrt{2}}\right)^2} = e^{-1/2} = \frac{1}{\sqrt{e}}$$

$$f\left(-\tfrac{1}{\sqrt{2}}\right) = e^{-\left(\frac{-1}{\sqrt{2}}\right)^2} = e^{-1/2} = \frac{1}{\sqrt{e}}$$

inflection points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$

approx: $(.7, .6)$ and $(-.7, .6)$

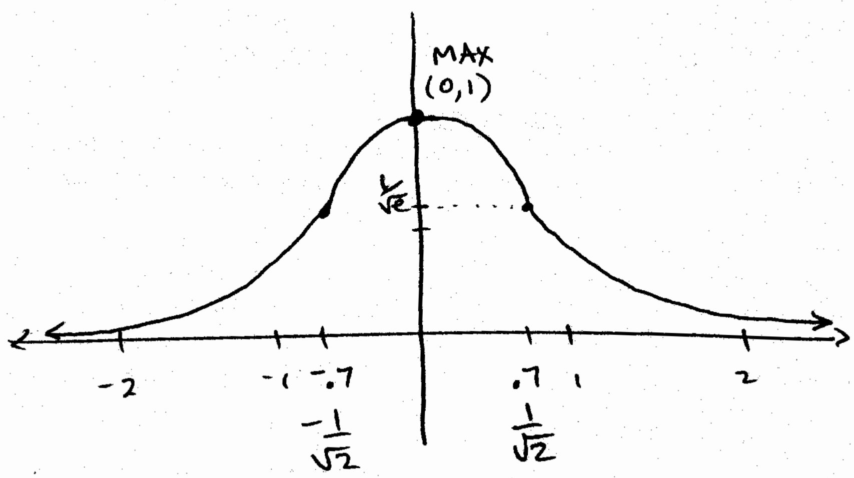
Putting it all together



$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$ $\left(+\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$

$(-.7, .6)$ $(+.7, .6)$

inflection inflection



x	y
0	1
-.7	.6
+.7	.6

(12) Graph using calculus.

$$f(x) = \ln(x^2+1)$$

$$f'(x) = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$

$$f''(x) = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$$

$$= \frac{2x^2+2-4x^2}{(x^2+1)^2}$$

$$= \frac{2-2x^2}{(x^2+1)^2} = \frac{2(1-x^2)}{(x^2+1)^2} = \frac{2(1-x)(1+x)}{(x^2+1)^2}$$

$f'(x)=0 \quad x=0 \quad$ no undef.

$f''(x)=0 \quad x=\pm 1 \quad$ no undef

f' : (−) | (−) | (+) | (+)

f'' : (−) | (+) | (+) | (−)

−1   0   1

inf IP   MIN   IP

$f(-1) = \ln((-1)^2+1) = \ln(1+1) = \ln(2) \quad (-1, \ln 2)$

$f(1) = \ln(2) \quad (1, \ln 2)$

$f(0) = \ln(0+1) = \ln(1) = 0 \quad (0,0)$